Partial Solutions to *Lectures on Holomorphic Curves in Symplectic and Contact Geometry* by Chris Wendl

Based on Version 3.3

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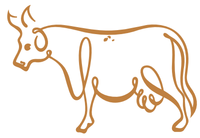

Contents

Introduction

Warm up: Holomorphic curves in $\mathbb{C}^n$

1.1.1. The usual CR equations are $i\frac{\partial u}{\partial x_j} = \frac{\partial u}{\partial y_j}$ for all $1 \leq j \leq m$, equivalently $i \circ du\left(\frac{\partial}{\partial x_j}\right) = du\left(\frac{\partial}{\partial y_j}\right)$ and we also we have that $\frac{\partial}{\partial y_j} = i\frac{\partial}{\partial x_j}$ with i applied as a linear map, thus $i \circ du = du \circ i$ over $\frac{\partial}{\partial x_j}$'s. Similarly multiplying the sides of the above CR equations by i gives $-du\left(\frac{\partial}{x_j}\right) = du\left(-\frac{\partial}{\partial x_j}\right) = idu\left(\frac{\partial}{y_j}\right)$ and noting that $-\frac{\partial}{\partial x_j} = i\frac{\partial}{y_j}$ we are seeing $du \circ i = idu$ over the $\frac{\partial}{\partial y_j}$'s. And the implication is reversible too.

1.1.2. To prove (1.1.2), in the easier case of $n = 1$, for vectors $X = x + ix'$ and $Y = y + iy'$ we have

$$\mathrm{Re}\langle i(x + ix'), y + iy'\rangle = -x'y + xy' = -dq_1 \otimes dp_1 + dp_1 \otimes dq_1 = dp_1 \wedge dq_1$$

and the $n \geq 2$ case is the coordinate-wise generalization of this. For the properties:

1. **Nondegeneracy:** if $V \neq 0$ then $\omega(V, iV) = \mathrm{Re}\langle iV, iV\rangle = \mathrm{Re}\|V\|^2 > 0$. So iV asserts that $\omega(V, \cdot)$ is nondegenerate.
2. **Closedness:** direct computation of the exterior derivative.

3. **The n-fold product:** in expanding the product $\omega_{\text{std}} \wedge \cdots \wedge \omega_{\text{std}}$, then only $2n$-forms that survive are the terms of form of the natural volume form. And when we are rearranging the wedges inside each of the remaining terms to get them to the form

$$dp_1 \wedge dq_1 \wedge dp_2 \wedge dq_2 \wedge \cdots \wedge dp_n \wedge dq_n,$$

we will be doing even number of wedge flips (a pair of flips for each pair dp_i, dq_i) therefore all terms appear with their own positive sign, overall giving a nonvanishing top form.

1.1.3. If ω is degenerate, then take $v \in \mathbb{R}^{2n}$ such that $\omega(v, w) = 0$ for all $w \in \mathbb{R}^{2n}$. We have that $\omega^n = \omega \wedge \omega^{n-1} = \frac{n!}{(n-1)!}\text{Alt}(\omega \otimes \omega^{n-1})$, and again recursively expanding ω^{n-1} by the alternation and keep doing this for all ω^k's, we see that overall ω^n is a bunch of $\underbrace{\omega \otimes \cdots \otimes \omega}_{n \text{ tensorands}}$ with some coefficients and signs; therefore if we plug in $w \in \mathbb{R}^{2n}$ alongside any $(n-1)$-tuple of vectors $v_1, \ldots, v_{n-1}$, each term in the expansion will vanish because in any term, because in each term, w appears in one of the tensorands and makes that tensorand vanish. This contradicts with ω^n being a volume form.

On the other hand, if ω is nondegenerate, pick a symplectic basis $\{e_i, f_i\}_{i=1}^n$ for $\mathbb{R}^{2n}$ (construction is discussed in Theorem 1.1 of Ana Cannas da Silva's book) with $\omega(e_i, f_j) = \delta_{ij}$, so $\omega = \sum_n de_i \wedge df_i$ and similar to problem 1.1.2 we get a volume form.

Hamiltonian systems and symplectic manifolds

1.2.1. The existence of X_H is guaranteed because of the nondegeneracy of ω. If X'_H is another such vector field we'd have $\omega_{std}(X_H - X'_H, \cdot) = 0$ but again as a nondegenerate ω identifies TM and T^*M, we have $X_H - X'_H = 0$. To confirm the form of X_H we compute,

$$\begin{aligned}\left(\sum_n dp_i \wedge dq_i\right)[X_H] &= \sum_n (dp_i \otimes dq_i - dq_i \otimes dp_i)\left[\frac{\partial H}{\partial p_i}\frac{\partial}{\partial q_i} - \frac{\partial H}{\partial q_i}\frac{\partial}{\partial p_i}\right] \\ &= \sum_n -\frac{\partial H}{\partial q_i}dq_i - \frac{\partial H}{\partial p_i}dp_i = -dH,\end{aligned}$$

as desired.

Some favorite examples

1.3.2. If we go from a coordinate system $(q_1, \ldots, q_n)$ to another one like $(\widetilde{q_1}, \ldots, \widetilde{q_n})$, then by the coordinate change we have,

$$\begin{aligned}\sum_i p_i \; dq_i = \sum_i p_i \sum_j \frac{\partial q_i}{\partial \widetilde{q_j}} \; d\widetilde{q_j} &= \sum_j \left(\sum_i \frac{\partial q_i}{\partial \widetilde{q_j}} \cdot p_i\right) \; d\widetilde{q_j} \\ &= \sum_j \widetilde{p_j} \; d\widetilde{q_j},\end{aligned}$$

and so the form of the form (!) doesn't depend on the coordinate system.

1.3.5. $\lambda_z(X)$ is real because $\mathrm{Im}(iz, X)$ is the real inner product of z with X as vectors in $\mathbb{R}^{2n+2}$, and hence equal to zero. Only the nondegeneracy condition is left. Directly computing the form, with $z = (p_1 + iq_1, \dots, p_{n+1} + iq_{n+1})$ and $X = (a_1 + ib_1, \dots, a_{n+1} + ib_{n+1})$, we have $\lambda_z(X) = \sum_k p_k\ b_k - q_k\ a_k$, i.e. $\lambda_z(X) = \sum_k p_k\ dq_k - q_k\ dp_k$ and so $d\lambda_z = \sum_k 2dp_k \wedge dq_k$, which is the same as the standard symplectic structure on $\mathbb{R}^{2n}$ at each point, and hence nondegenerate by exercise 1.1.2.

1.3.6. Same as the proof for $\mathbb{RP}^n$ being a smooth manifold (can be found e.g. in Lee's smooth manifolds book), and just substituting with $\mathbb{CP}^n$ and being a complex manifold.

Darboux's theorem and the Moser deformation trick

1.4.2. Theorem 1.1 of da Silva's notes.

1.4.7. By the cohomological hypothesis we can assume that $\omega_t = \omega_0 + d\lambda_t$. Therefore as in equation (1.4.1) we get $\mathcal{L}_{Y_t}\omega_t + \frac{d}{dt} d\lambda_t = d\iota_{Y_t}\omega_t + d\dot{\lambda}_t = 0$ and so it suffices to solve for $\omega_t(Y_t, \cdot) = \dot{\lambda}_t$. Then the isotopy generated by Y_t exists because of the compactness condition on M and we have $\frac{d}{dt}(\varphi_t^*\omega_t) = 0$, as desired.

1.4.8. We know that $H^2_{dR}(\mathbb{CP}^n) = \mathbb{R}$. So first, we notice that ω, which is an arbitrary symplectic form, is non-exact: if otherwise then $\omega = d\lambda$ for some 1-form λ. Since ω is closed, by induction and using the Leibniz rule on differential forms, ω^k's are also closed for all k, and therefore by another application of the Leibniz rule we have $\omega^n = d(\lambda \wedge \omega^{n-1})$. Therefore Stokes theorem gives $\int_{\mathbb{CP}^n} \omega^n = \int_{\partial\mathbb{CP}^n} \lambda \wedge \omega^{n-1} = 0$ but this is a contradiction because ω being symplectic implies ω^n being a volume form.

Therefore $[\omega]$ spans H^2 and $[\omega'] = c[\omega] = [c\omega]$, and $c \neq 0$ since ω' is also nondegenerate. Moser's stability theorem then implies that ω' and $c\omega$ are isotopic. That $c > 0$ follows from the existence of the deformation equivalence ω_t being a continuous family of nondegenerate forms interpolating $c \cdot \omega$ and $+1 \cdot \omega$.

From symplectic geometry to symplectic topology

No problems in this section.

Contact geoemtry and the Weinstein conjecture

Contact manifolds skipped in this first reading.

Symplectic fillings of contact manifolds

Contact manifolds skipped in this first reading.

Fundamentals

Almost complex manifolds and J-holomorphic curves

2.1.1. (a) Take a vector $x_1 \neq 0$ and define $y_1 := Jx_1$. Then for each $2 \leq k \leq n$ let $x_k \in V \setminus \operatorname{span}(x_1, y_1, \ldots, x_{k-1}, y_{k-1})$ and again set $y_k := Jx_k$. By construction, they form a basis and J has the matrix form $\mathbb{J}_{std}$ in this basis.

(b) If there was just a J then $\det(-J) = (-1)^n \det(J) = -\det(J)$. Note also that $J^2 = -1$ implies $\det(J) = \pm 1$; which in any case the first sentence implies $\det(-J)\det(J) = -1$, but this cannot hold because also $\det(-J)\det(J) = \det(-J^2) = \det(1) = 1$, a contradiction.

(c) Note that such a map A would be complex linear and in that case $\det_{\mathbb{R}} A = |\det_{\mathbb{C}} A|^2 > 0$.

2.1.2. (a) If we take a trivialization of $\pi : E^{2k+n} \to M^n$, i.e. an open $U \ni p$ and $\varphi : \pi^{-1}(U) \to \mathbb{R}^n \times \mathbb{R}^{2k}$, we can then define a bundle automorphism $\psi : \mathbb{R}^n \times \mathbb{R}^{2k} \to \mathbb{R}^n \times \mathbb{R}^{2k}$ given by $(q, v) \mapsto (q, A_q(v))$ where $A_q \in \mathrm{GL}(2n, \mathbb{R})$ is the matrix that takes the basis $\{x_i, y_i\}_{i=1}^k \subset \mathbb{R}^{2k}$ as defined in exercise 2.1.1, to the basis

$$\{(0, \ldots, 0, \underbrace{1}_{2i\text{-th place}}, 0, \ldots, 0), (0, \ldots, 0, \underbrace{1}_{(2i+1)\text{-th place}}, 0, \ldots, 0)\} \subset \mathbb{R}^{2k}.$$

Then $\psi \circ \varphi$ is a desirable trivialization.

(b) The transition maps between say trivialization φ and ψ, fiber-wise amount to linear change of coordinate maps $A_q : \mathbb{R}^{2k} \to \mathbb{R}^{2k}$ for all $q \in U$. Let $\{x_i, y_i\}_{i=1}^{2k}$ and $\{\tilde{x}_i, \tilde{y}_i\}_{i=1}^{2k}$ respectively be the two bases afforded by φ and ψ; then $[A_q]_{ij} = d\tilde{e}_j(e_i)$. Noting that by linearity of J we have $dJ = J$, we can write

$$\begin{aligned}
JA(x_1) &= J \sum_i d\tilde{x}_i(x_1)\tilde{x}_i + d\tilde{y}_i(x_1)\tilde{y}_i \\
&= \sum_i d\tilde{x}_i(x_1)\tilde{y}_i - d\tilde{y}_i(x_1)\tilde{x}_i \\
&= \sum_i d\tilde{x}_i(J(-y_1))\tilde{y}_i - d\tilde{y}_i(J(-y_1))\tilde{x}_i \\
&= \sum_i -d(\tilde{x}_i \circ J)(y_1)\tilde{y}_i + d(\tilde{y}_i \circ J)(y_1)\tilde{x}_i \\
&= \sum_i - - d\tilde{y}_i(y_1)\tilde{y}_i + d\tilde{x}_i(y_1)\tilde{x}_i \\
&= A(y_1) = AJ(x_1).
\end{aligned}$$

And similarly for the rest of x_j's.

2.1.5. (a) By the tensor characterization lemma (Lee 2nd Ed. lemma 12.24) a map $\mathfrak{X}(M) \times \mathfrak{X}(M) \to C^\infty(M)$ is induced by a smooth tensor field if and only if it is multilinear over $C^\infty(M)$, which can be directly verified here.

(b) At each point $p \in M$ and for any $0 \neq v \in T_pM$ we can directly verify that

$$N_J(v, v) = N_J(Jv, v) = N_J(v, Jv) = N_J(Jv, Jv) = 0,$$

but since M is 2-dimensional, the set $\{v, Jv\}$ forms a basis for T_pM and implies that N_J vanishes at every p.

(c) For each point p, let X^i and Y^i be vector fields such that $X^i_p = \frac{\partial}{\partial x^i}, Y^i_p = \frac{\partial}{\partial y^i}$ where $\{x_i, y_i\}_{i=1}^n$ is a complex coordinate coming from the complex structure, i.e. $J\frac{\partial}{\partial x^i} = \frac{\partial}{\partial y^i}$. Note that

$$[X^i, X^j]_p = [Y^i, Y^j]_p = [X^i, Y^j]_p = 0$$

for all $1 \leq i, j \leq n$, because the tangents come from coordinate vectors. We can then easily see that these imply

$$N_J(X^i, X^j) = N_J(X^i, Y^j) = N_J(Y^i, X^j) = N_J(Y^i, Y^j) = 0,$$

which in turn vanishes N_J at each p.

Compatible and tame almost complex structures

2.2.1. Note that $\mathrm{GL}(n, \mathbb{C})$ matrices are the ones like A that satisfy $Ai - iA = 0$, a closed condition, hence $\mathrm{GL}(n, \mathbb{C})$ is a closed subgroup and the quotient $\mathrm{GL}(2n, \mathbb{R})/\mathrm{GL}(n, \mathbb{C})$ is a smooth manifold of dimension $(2n)^2 - 2 \cdot n^2 = 2n^2$. The map Φ is clearly smooth and for descendance of the map, we see that for any $A \in \mathrm{GL}(2n, \mathbb{R})$ and $C \in \mathrm{GL}(n, \mathbb{C})$ we have $\Phi(AC) = ACiC^{-1}A^{-1} = AiCC^{-1}A^{-1} = \Phi(A)$. Also $\mathrm{GL}(2n, \mathbb{R})/\mathrm{GL}(n, \mathbb{C})$ is non-compact because for example for the family of representative matrices A_t with $A_t((1, \vec{\mathbf{0}}_{2n-1})) = (1, \vec{\mathbf{0}}_{2n-1})$ and $A_t(0, 1, \vec{\mathbf{0}}_{2n-2}) = t \cdot (0, 1, \vec{\mathbf{0}}_{2n-2})$, we have $A_t i A_t^{-1}(1, \vec{\mathbf{0}}_{2n-1}) = t \cdot (0, 1, \vec{\mathbf{0}}_{2n-2})$, therefore $[A_t]_{1,2}$ grows unboundedly if $t \to \infty$.

To see the injectivity, if $\Phi(A) = \Phi(B)$, then $B^{-1}Ai = iB^{-1}A$, i.e. $B \simeq A \mod \mathrm{GL}(n, \mathbb{C})$. To see that the image is precisely $\mathcal{J}(\mathbb{C}^n)$, firstly we have $(AiA^{-1})^2 = AiA^{-1}AiA^{-1} = -1$ and so $\operatorname{Im}\Phi \subset \mathcal{J}(\mathbb{C}^n)$. Secondly, if $J \in \mathcal{J}(\mathbb{C}^n)$, then pick a complex basis $\{x_k, y_k\}_{k=1}^n$ corresponding to J, i.e. such that $J(x_k) = y_k$. Also let $\{1_k, i_k\}_{k=1}^n$ be the standard basis of $\mathbb{R}^{2n}$. If we define $A \in \mathrm{GL}(2n, \mathbb{R})$ by $A(1_k) = x_k$ and $A(i_k) = y_k$ we can see that $AiA^{-1}(x_k) = Ai(1) = A(i) = y_k = J(x_k)$, and therefore Φ is also surjective. The tangent space at each point $J \in \mathcal{J}(\mathbb{C}^n)$ is by definition equal to the quotient of vector spaces $T_J\mathrm{GL}(2n, \mathbb{R})/T_J\mathrm{GL}(n, \mathbb{C})$. We know $T_J\mathrm{GL}(2n, \mathbb{R}) = \mathrm{End}(2n, \mathbb{R})$, and note that every element $A \in \mathrm{End}(2n, \mathbb{R})$ can be written as the sum of a complex linear and a complex antilinear map as the following. Say if $n = 1$ and $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then let $B = \begin{pmatrix} \frac{a+d}{2} & \frac{b-c}{2} \\ -\frac{b-c}{2} & \frac{a+d}{2} \end{pmatrix} \in \mathrm{End}_{\mathbb{C}}(\mathbb{C}^n)$ and $C = \begin{pmatrix} \frac{a-d}{2} & \frac{b+c}{2} \\ \frac{b+c}{2} & -\frac{a-d}{2} \end{pmatrix} \in \overline{\mathrm{End}}_{\mathbb{C}}(\mathbb{C}^n)$, then $A = B + C$. For bigger n's just do this construction per each complex coordinate. Considering also the dimensions, we can confirm that $\mathrm{End}_{\mathbb{R}}(\mathbb{C}^n) = \mathrm{End}_{\mathbb{C}}(\mathbb{C}^n) \oplus \overline{\mathrm{End}}_{\mathbb{C}}(\mathbb{C}^n)$ and so the equality for the tangent space follows as well.

2.2.2. Fixing some complex structure J_0 and using the atlas in which J_0 is given by the matrix $\mathbb{J}_{std}$ in all trivialization (using problem 2.1.2), define the function $f : \mathcal{J}(E) \to \Gamma(\mathrm{Aut}_{\mathbb{R}}(E)/\mathrm{Aut}_{\mathbb{C}}(E, J_0) \to M)$ with $f(J)$ at $p \in M$ being the class of the change of coordinate matrix A_p such that $A_p\mathbb{J}_{std}A_p^{-1} = [J]_p$ (such A_p's are afforded by taking a J-complex basis $\{x_k, y_k\}$ to the standard complex basis $\{1_k, i_k\}$). Furthermore for any section $A_p \in \Gamma(\mathrm{Aut}_{\mathbb{R}}(E)/\mathrm{Aut}_{\mathbb{C}}(E, J_0) \to M)$, let J_p defined via A_p as before, and the fact that A_p is smooth in p implies that so is J_p, giving a complex structure over the whole $E \to M$. Thus we have a bijection.

2.2.3. Considering the first order approximations, we are interested in $J_{\epsilon H}$ for any $H \in \overline{\text{End}}_{\mathbb{C}}(\mathbb{C}^n)$ and $\epsilon \to 0$, and we see that,

$$\begin{aligned} J_{\epsilon H} &= \left(1 + \frac{1}{2} i \epsilon H\right) i \left(1 + \frac{1}{2} i \epsilon H\right)^{-1} \\ &\simeq \left(1 + \frac{1}{2} i \epsilon H\right) i \left(1 - \frac{1}{2} i \epsilon H\right) \\ &= i \left(1 - \frac{1}{2} i \epsilon H\right)^2 \\ &\simeq i \left(1 - 2\frac{1}{2} i \epsilon H\right) \\ &= i + \epsilon H = J_0 + \epsilon H, \end{aligned}$$

where for the second line we used the geometric expansion of the last term, and for the third line we used anti-linearity of H. This it seems to be the justification for the modification in the parametrization (2.2.1) of the book.

2.2.5. If ω is compatible, then

$$\omega(Jv, Jw) = g_J(Jv, w) = g_J(w, Jv) = \omega(w, -v) = \omega(v, w).$$

Conversely if ω is J-invariant, then

$$\omega(v, Jw) = \omega(Jv, J^2 w) = \omega(Jv, -w) = \omega(w, Jv),$$

so ω is symmetric and hence compatible.

2.2.6. For tameness, we see that for all $f \in F$ and $d \in F^{\perp\omega}$ with $f + d \neq 0$ we have

$$\omega(f + d, (j \oplus j')(f + d)) = \omega(f, j(f)) + \omega(d, j'(d)) > 0$$

where the two other terms vanish because of the symplectic complementation, and the inequality because of tameness of j and j'.

For compatibility, similarly we have

$$\begin{aligned} \omega(f_1 + d_1, (j \oplus j')(f_2 + d_2)) &= \omega(f_1, j(f_2)) + \omega(d_1, j'(d_2)) \\ &= \omega(f_2, j(f_1)) + \omega(d_2, j'(d_1)) \\ &= \omega(f_2 + d_2, (j \oplus j')(f_1 + d_1)), \end{aligned}$$

where we used compatibility of j and j' for the second line and symplectic complementation for the last line.

2.2.7. Existence of such a system of trivializations clearly implies compatibility of J with ω. Conversely, as in the hint we can define such a Hermitian metric $\langle v, w \rangle$ for the complex vector bundle (E, J), and as in exercise 2.1.2 consider an atlas of trivializations like $\varphi : \pi^{-1}(U) \to \mathbb{R}^m \times \mathbb{R}^{2n}$ for which $[J_q] = \mathbb{J}_{std}$ for each $q \in \mathbb{R}^{2n}$, so we also have had that transition maps are complex linear. Then my idea is similar to 2.1.2: The map $\text{GL}(n, \mathbb{C}) \to \mathcal{H}(\mathbb{C}^n)$ defined from nonsingular matrices to Hermitian matrices (i.e. the space of Hermitian inner products) given by $C \mapsto C^T \cdot \overline{C} = \langle C(\ ), C(\ ) \rangle_{std}$ is a surjective smooth submersion (by direct computation of the derivative). Therefore locally if we look at the inner product $[\langle\cdot,\cdot\rangle]_q = H_q \in \mathcal{H}(\mathbb{C}^n)$, we can define a smooth section of the previous map by $\sigma : \mathcal{H}(\mathbb{C}^n) \to \mathrm{GL}(n,\mathbb{C})$ and we can define a bundle complex isomorphism (so $\mathbb{J}_{std}$ remains unchanged) by $\psi : \mathbb{R}^m \times \mathbb{R}^{2n} \to \mathbb{R}^m \times \mathbb{R}^{2n}$ via $(q,v) \mapsto (q, \sigma(H_q)^{-1}(v))$ which sends all H_q's to the standard Hermitian inner product $\langle\cdot,\cdot\rangle_{std}$. Hence in the trivialization $\psi \circ \varphi$ the symplectic structure is also the standard one because $\mathrm{Im}(\langle\cdot,\cdot\rangle_{std}) = \omega_{std}$, as desired.

2.2.9. For non-emptiness, as indicated in the hint, Hermitian metrics for the complex vector bundle (E,J) are sections of the Hermitian vector bundle over the base manifold, and hence they exist. Taking the imaginary part provides a J-compatible symplectic form, therefore both of $\Omega^\tau(E,J)$ and $\Omega(E,J)$ are non-empty. Convexity also follows from the fact that if $v \neq 0$ then

$$\big((1-t)\omega_0 + t\omega_1\big)(v,Jv) = (1-t)\omega_0(v,Jv) + t\omega_1(v,Jv) > 0$$

because $\omega_0(v,Jv), \omega_1(v,Jv) > 0$. And also that symmetricity is preserved by linear combination of symmetric forms.
